International Journal of Engineering Research and Modern Education (IJERME)
Impact Factor: 6.525, ISSN (Online): 2455 - 4200
(www.rdmodernresearch.com) Volume 2, Issue 1, 2017

NUMERICAL SOLUTION OF SECOND ORDER FUZZY DIFFERENTIAL EQUATIONS BY ADOMIAN DECOMPOSITION METHOD

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Cite This Article: S. Lakshmi Narayanan, "Numerical Solution of Second Order Fuzzy Differential Equations by Adomian Decomposition Method", International Journal of Engineering Research and Modern Education, Volume 2, Issue 1, Page Number 58-64, 2017.

Abstract:

The present paper studies the numerical method for nonlinear Fuzzy Differential Equations by Adomian decomposition method and, we present an example with initial condition having two different solutions to illustrate the efficiency of the proposed Adomian method.

Key Words: Linear Fuzzy Differential Equation, Fuzzy Initial Value Problem & Adomian Decomposition Method

1. Introduction:

Fuzzy differential equations play an important role in an increasing number of systems models in biology, engineering, physics and other sciences. For example, in population models, civil engineering, bioinformatics and computational biology, quantum optics and gravity and in modeling hydraulic and second order linear fuzzy differential equations are one of the simplest fuzzy differential equations which may appear in many applications [5]. Adomian decomposition method (ADM) has proven successfully for getting analytical solutions of linear and nonlinear differential equations by providing solutions in terms of convergent power series [1, 2, 3]. The advantage of the method is that it can be applied for all types of integral equations, linear or nonlinear, homogeneous or non-homogeneous with constant co-efficient or with variable coefficient [6, 7, 8]. And also this method is capable of greatly reducing the size of computation work with high accuracy of numerical solution [12, 13, 14]. In this Section, The second order linear fuzzy differential equation is solved by Fuzzy Adomian Decomposition Method (FADM) We replace the initial problem by its parametric form and then solve the new system which consist of two classic ordinary differential equations with initial conditions, then check to see whether this solution define a fuzzy function.

2. Preliminaries:

Definition 1 [6]: A fuzzy number is a map $u:\mathbb{R} \to I=[0,1]$ which satisfies

i. u is upper semi continuous,
ii. u(x) = 0 outside some interval $[c,d]\subset\mathbb{R}$
iii. There exist real numbers a, b such that $c \le a \le b \le d$

Where

a. u(x) is monotonic increasing on [c, a],
b. u(x) is monotonic decreasing on [b, d],
c. u(x) = 1, $a \le x \le b$

The set of all such fuzzy numbers is represented by E^1.

Definition 2 [8]: An arbitrary fuzzy number in parametric form is represented by an ordered pair functions $\left(\underline{u}(r),\overline{u}(r)\right)$, $0 \le r \le 1$, which satisfy the following requirements:

1. $\underline{u}(r)$ is a bounded left-continuous non-decreasing function over [0, 1]
2. $\overline{u}(r)$ is a bounded left -continuous non -increasing function over [0, 1]
3. $\underline{u}(r) \le \overline{u}(r),\ 0 \le r \le 1$

A crisp number α is simply represented by $\underline{u}(r)=\overline{u}(r)=\alpha,\ 0\le r\le 1$ for arbitrary $u=\left(\underline{u}(r),\overline{u}(r)\right), v=\left(\underline{v}(r),\overline{v}(r)\right)$ and $k\in\mathbb{R}$, we define equality, addition and multiplication by *k* as

1. u=v if and only if $\underline{u}(r)=\underline{v}(r)$ and $\overline{u}(r)=\overline{v}(r)$
2. $u+v=\left(\underline{u}(r)+\underline{v}(r),\overline{u}(r)+\overline{v}(r)\right)$
3. $ku=\begin{cases}(k\underline{u},k\overline{u}), & k\ge 0,\\ (k\overline{u},k\underline{u}), & k<0.\end{cases}$

Definition 3 [5]: The Hausdorff distance of two fuzzy numbers given by $D: \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+ \cup \{0\}$ is defined as follows: $D(u,v) = \sup_{r\in[0,1]} \max\left\{\left|u_-^r - v_-^r\right|, \left|u_+^r - v_+^r\right|\right\} = \sup_{r\in[0,1]}\left\{d_H([u]^r,[v]^r)\right\}$,

Where $[u]^r = [u_-^r, u_+^r], [v]^r = [v_-^r, v_+^r]$. We denote $\|.\| = D(.,0)$.

Definition 4 [11]: Let I be a real interval. A mapping $x: I \to E$ is called a fuzzy process and its *r*-level set is denoted by $[x(t)]_r = [x_1(t;r), x_2(t;r)],\ t \in I, r \in (0,1]$ the derivative $x'(t)$ of a fuzzy process x is defined by $[x'(t)]_r = [x_1'(t;r), x_2'(t;r)],\ t \in I, r \in (0,1]$.

3. Fuzzy Differential Equations:

Let us consider the second-order fuzzy ordinary differential equations of the form

$$\begin{cases} x''(t) = f(t,x,x') \\ x(t_0) = x_0, \quad x'(t_0) = x_0' \end{cases} \tag{1}$$

where *x* is a fuzzy function of *t*, *f(x, t)* is a fuzzy function of the crisp variable *t* and the fuzzy derivative of *x* and $x(t_0) = x_0$ and $x'(t_0) = x_0'$ is a triangular shaped fuzzy number. We denote the fuzzy function *x* by $x = [\underline{x}, \overline{x}]$. The α level set of *x (t)* is defined as $[x(t)]_r = [\underline{x}(t,\alpha), \overline{x}(t,\alpha)]$ and $[x(t)]_r = [\underline{x}(t_0,\alpha), \overline{x}(t_0,\alpha)],\ \alpha \in [0,1]$ (2)

Consider the differential equation written in the following form $Lu + Ru + Nu = g$ (3)

where L is linear operator, N is a non-linear operator, and *g(x)* is a inhomogeneous term. Then, we can construct a correct functional as follows; where L is a linear operator, N is a nonlinear operator and *g(x)* is an inhomogeneous term. If differential equations describes by *n* order, where the differential operator L is given by Adomian (1988) and $L(.) = \frac{d^n(.)}{dt^n}$ the inverse operator L^{-1} is therefore considered a n-fold integral operator defined by $L^{-1}(*) = \int_0^x\int_0^x \ldots \int_0^x (*)dx \ldots dx$, then

$$x = L^{-1}(*)(g(t) - Nx) \tag{4}$$

The Adomian technique consists of the solution of (3) as an infinite series

$$x = \sum_{n=0}^{\infty} x_n(t) \tag{5}$$

and decomposing the nonlinear operator N as

$$Nx = \sum_{n=0}^{\infty} A_n(t) \tag{6}$$

where An are Adomian polynomials [1] of $x_0, x_1, x_2, \ldots, x_n$

$$A_n = \frac{1}{n}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i x_i\right)\right]_{\lambda=0} \tag{7}$$

n = 0, 1, 2 . . . substituting the derivatives (4), (5) and (6) which gives $\sum_{n=0}^{\infty} x_n(t) = L^{-1}(g(t)) - L^{-1}\left(\sum_{n=0}^{\infty} A_n\right)$ (8)

thus $x_0 = L^{-1}(g(t))$

$$x_{n+1} = L^{-1}\left(\sum_{n=0}^{\infty} A_n\right) = A_n(x_0, x_1, \ldots, x_n) \tag{9}$$

n = 0, 1, 2 ... We then define the *k*-term approximate to the solution *x* by

$$\Phi_k[x] = \sum_{n=0}^{\infty} x_n \ \ and \ \ \lim_{k\to\infty} \Phi_k[x] = x \tag{10}$$

Practical formula for the calculation of Adomian decomposition polynomials are given in A_n. How-ever all term of the series cannot be determined usually $A_n = \sum_{n=0}^{\infty} x_n$ is approximated with truncated series $\Phi_k = x_0 + x_1 + x_2 \ldots + x_{n-1}$.

Example 1: Consider the following fuzzy differential equation with fuzzy initial value problem:

$$\begin{cases} x'' - 4x' + 4x = 0\ ,\ t \in [0,1] \\ x(0) = (2+r, 4-r), x'(0) = (5+r, 7-r) \end{cases} \tag{11}$$

where the exact solution is as follow:

$\underline{x}(t)=(2+r)e^{2t}+(1-r)te^{2t}$

$\overline{x}(t)=(4-r)e^{2t}+(r-1)te^{2t}$

We can apply Adomian decomposition method to find the approximate solution.
Consider

$L\overline{x}(t)=4\overline{x}\,'(t)-4\overline{x}(t)$

$L\underline{x}(t)=4\underline{x}\,'(t)-4\underline{x}(t)$

With initial conditions

$x(0)=(2+r,4-r)\Rightarrow \overline{x}(0)=2+r,\underline{x}(0)=4-r$

$x'(0)=(5+r,7-r)\Rightarrow \overline{x}\,'(0)=5+r,\underline{x}\,'(0)=7-r$

Let $L=\frac{d^2}{dt^2}$ is the differential operator then the inverse operator $L^{-1}(.)=\int_0^t\int_0^t(.)dtdt$ and using the given initial conditions we have

$\overline{x}(t,r)=(2+r)+t(5+r)+4\int_0^t\int_0^t(\overline{x}\,'(t)-\overline{x}(t))dtdt$

$\underline{x}(t,r)=(4-r)+t(7-r)+4\int_0^t\int_0^t(\underline{x}\,'(t)-\underline{x}(t))dtdt$

on applying the Adomian decomposition method, we get

$\overline{x}_0(t,r)=(2+r)+t(5+r)$

$\overline{x}_{n+1}(t,r)=4\int_0^t\int_0^t(\overline{x}_n'(t)-\overline{x}_n(t))dtdt$

n=0, 1, 2…

$\overline{x}_1(t,r)=6t^2-t^3\left(\frac{2r+10}{3}\right)$

$\overline{x}_2(t,r)=8t^3-\left(\frac{2r+16}{3}\right)t^4+\left(\frac{2r+10}{15}\right)t^5$

The approximate solution of $\overline{x}(t,r)$ as follows

$$\overline{\phi}_n(t,r)=\sum_{i=0}^{n}\overline{x}_i(t,r) \qquad (12)$$

$$=(2+r)+t(5+r)+6t^2-t^3\left(\frac{2r+10}{3}\right)+8t^3-\left(\frac{2r+16}{3}\right)t^4+\left(\frac{2r+10}{15}\right)t^5+\cdots\cdots$$

and

$\underline{x}_0(t,r)=(4-r)+t(7-r)$

$\underline{x}_{n+1}(t,r)=4\int_0^t\int_0^t(\underline{x}_n'(t)-\underline{x}_n(t))dtdt$

n=0, 1, 2…

$\underline{x}_1(t,r)=6t^2-t^3\left(\frac{2r-14}{3}\right)$

$\underline{x}_2(t,r)=8t^3+\left(\frac{2r-20}{3}\right)t^4-\left(\frac{2r-14}{15}\right)t^5$

$\vdots$

The approximate solution of $\underline{x}(t,r)$ as follows

$$\underline{\phi}_n(t,r)=\sum_{i=0}^{n}\underline{x}_i(t,r) \qquad (13)$$

$$=(4-r)+t(7-r)+6t^2-t^3\left(\frac{2r-14}{3}\right)+8t^3+\left(\frac{2r-20}{3}\right)t^4-\left(\frac{2r-14}{15}\right)t^5+\cdots\cdots$$

Which are approximations for equation (12) and (13) respectively we can obtain only an approximate solution. The results obtained by ϕ_{10} term of Fuzzy Adomian decomposition method (equation (12) and (13)) are compared with exact solutions [3] in Fig. 1.

Table 1: Comparison between the exact solution and the approximate solution for t =0.1

r	Present Result		Exact Result [4]	
	$\overline{x}(t,r)$	$\underline{x}(t,r)$	$\overline{x}(t,r)$	$\underline{x}(t,r)$
0	4.7635	2.5649	4.7635	2.5649
0.1	4.6535	2.6749	4.6535	2.6749
0.2	4.5436	2.7848	4.5436	2.7848
0.3	4.4337	2.8947	4.4337	2.8947
0.4	4.3238	3.0047	4.3238	3.0047
0.5	4.2138	3.1146	4.2138	3.1146
0.6	4.1039	3.2245	4.1039	3.2245
0.7	3.9940	3.3344	3.9940	3.3344
0.8	3.8841	3.4444	3.8841	3.4444
0.9	3.7741	3.5543	3.7741	3.5543
1	3.6642	3.6642	3.6642	3.6642

Table 2: Comparison between the exact solution and ADM solution for t =0.5

r	Present Result		Exact Result [4]	
	$\overline{x}(t,r)$	$\underline{x}(t,r)$	$\overline{x}(t,r)$	$\underline{x}(t,r)$
0	9.5103	6.7918	9.5140	6.7957
0.1	9.3744	6.9278	9.3781	6.9316
0.2	9.2384	7.0637	9.2422	7.0675
0.3	9.1025	7.1996	9.1062	7.2034
0.4	8.9666	7.3355	8.9703	7.3394
0.5	8.8307	7.4714	8.8344	7.4753
0.6	8.6947	7.6074	8.6985	7.6112
0.7	8.5588	7.7433	8.5626	7.7471
0.8	8.4229	7.8792	8.4267	7.8830
0.9	8.2870	8.0151	8.2908	8.0189
1	8.1511	8.1511	8.1548	8.1548

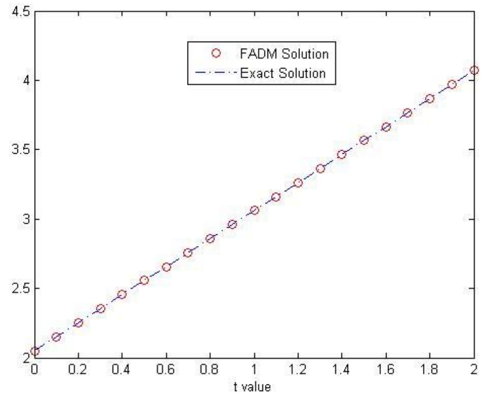

Figure 1: Solution of $\overline{x}(t,r)$ for various values of r at t=0.02 using FADM with Exact

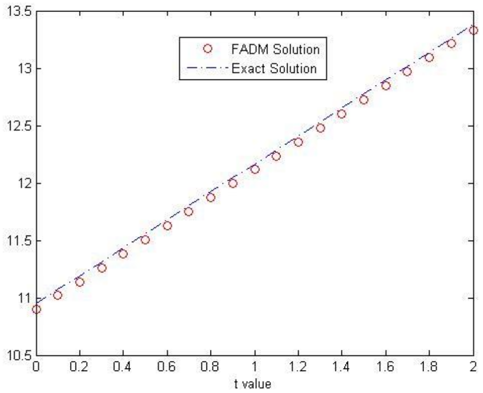

Figure 2: Solution of $\overline{x}(t,r)$ for various values of r at t=0.7 using FADM with Exact

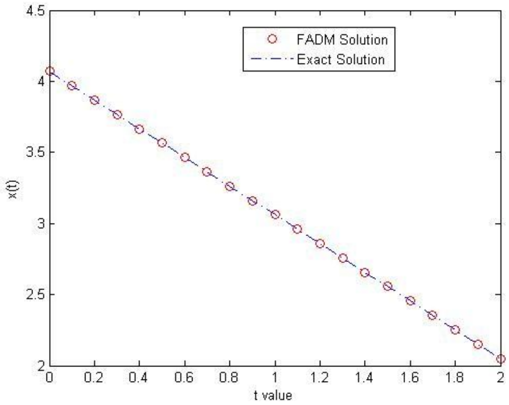

Figure 3: Solution of $\underline{x}(t,r)$ for various values of r at t=0.02 using FADM with Exact

Example 2: Consider the following fuzzy differential equation with fuzzy initial value problem

$$\begin{cases} x''-2x'=0, & t\in[0,0.5] \\ x(0)=(r,2-r), & x'(0)=(3+r,4) \end{cases} \tag{14}$$

where the exact solution is as follow

$\underline{x}(t)=\frac{r}{2}-\frac{3}{2}+\frac{3+r}{2}e^{2t}$ and $\overline{x}(t)=-r+2e^{2t}$

We can apply Adomian decomposition method to find the approximate solution.

Consider

$L\overline{x}(t)=2\overline{x}'(t)$

$L\underline{x}(t)=2\underline{x}'(t)$

With initial conditions

$x(0)=(r,2-r)\Rightarrow\overline{x}(0)=r,\underline{x}(0)=2-r$

$x'(0)=(3+r,4)\Rightarrow\overline{x}'(0)=3+r,\underline{x}'(0)=4$

Let $L=\frac{d^2}{dt^2}$ is the differential operator then the inverse operator $L^{-1}(.)=\int_0^t\int_0^t(.)dtdt$ and using the given initial conditions we have $\overline{x}(t,r)=(3+r)+rt+2\int_0^t\int_0^t\left(\overline{x}'(t)\right)dtdt$

$\underline{x}(t,r)=(2-r)+4t+2\int_0^t\int_0^t\left(\underline{x}'(t)\right)dtdt$

on applying the Adomian decomposition method,

$\overline{x}_0(t,r)=(3+r)+rt$

$\overline{x}_{n+1}(t,r)=2\int_0^t\int_0^t\left(\overline{x}_n'(t)\right)dtdt$

n=0, 1, 2…

$\overline{x}_1(t,r)=rt^2$

$\overline{x}_2(t,r)=\frac{2rt^3}{3}$

⋮

The approximate solution of $\overline{x}(t,r)$ as follows

$$\overline{\phi}_n(t,r)=\sum_{i=0}^{n}\overline{x}_i(t,r) \tag{15}$$

$$=3+r+rt+rt^2+\frac{2rt^3}{3}+\frac{rt^4}{3}+\frac{2rt^5}{15}+\frac{2rt^6}{45}+\frac{4rt^7}{315}+\cdots$$

And $\underline{x}_0(t,r)=4t-r+3$

$\underline{x}_{n+1}(t,r)=2\int_0^t\int_0^t\left(\underline{x}_n'(t)\right)dtdt$

n=0, 1, 2…

$\underline{x}_1(t,r)=4t^2$

$\underline{x}_2(t,r)=\dfrac{8t^3}{3}$

$\vdots$

The approximate solution of $\underline{x}(t,r)$ as follows

$$\underline{\phi}_n(t,r)=\sum_{i=0}^{n}\underline{x}_i(t,r) \qquad (16)$$

$$=3-r+4t+4t^2+\frac{8t^3}{3}+\frac{4t^4}{3}+\frac{8t^5}{15}+\frac{8t^6}{45}+\frac{16t^7}{315}+\cdots$$

which are approximations for equation (15) and (16) respectively we can obtain only an approximate solution. The results obtained by ϕ_{10} term of Fuzzy Adomian decomposition method (equation (15) and (16)) are compared with exact solutions [3].

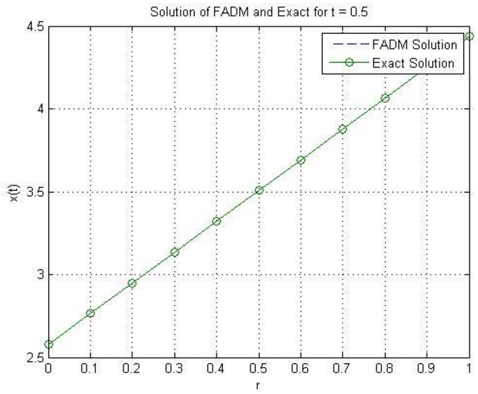

Figure 5: Solution of $\underline{x}(t,r)$ for various values of r at t=0.5 using FADM with Exact in example 2

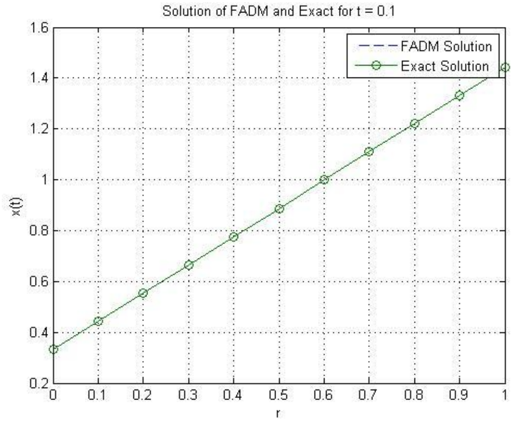

Figure 6: Solution of $\underline{x}(t,r)$ for various values of r at t=0.1 using FADM with Exact in example 2

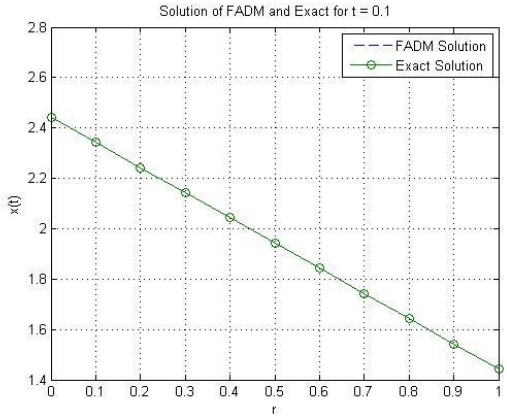

Figure 7: Solution of $\overline{x}(t,r)$ for various values of r at t=0.1 using FADM with Exact in example 2

4. Conclusion:

Adomian decomposition method is a powerful technique which is capable of handling higher order fuzzy differential equations. The methods have been successfully employed to higher order fuzzy differential equations. When the approximation results are found by using Adomian decomposition method and compared to The Exact Solutions With Existing Results.

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