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A STUDY ON MHD BOUNDARY LAYER FLOW OVER A NON-LINEAR STRETCHING SHEET USING MODIFIED HOMOTOPY ANALYSIS METHOD

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Abstract:

In this study a problem of the MHD boundary layer flow of an incompressible viscous fluid over a nonlinear stretching sheet is considered. Governing non-linear partial differential equations are transformed to non-linear ordinary differential equations using similarity transformation. A strong analytic tool for nonlinear problems, the Modified Homotopy analysis method is employed to obtain the analytical solution for the nonlinear differential equation. Graphical results of fluid velocity have been presented and discussed for the related parameters.

Key Words: MHD, Viscous Flow, Stretching Sheet & Modified HAM.

1. Introduction:

The study of boundary layer flows due to a stretching sheet is very important as it finds many practical applications in chemical and metallurgy industries. It was first studied by Sakiadis [1]. Crane [2] analyzed the flow over a linearly stretching sheet for the steady two dimensional problems. These types of flows usually occur in the drawing of plastic films and artificial fibers. Viscous flows due to a moving sheet in electromagnetic fields i.e. magneto hydrodynamic (MHD) flows have various applications in the areas of engineering and technology such as MHD power generator, MHD flow meters, MHD pumps, etc. MHD flows of Newtonian fluids were investigated by Pavlov [3], Chakrapati and Gupta [4], Vajravelu [5], etc.

For the governing non-linear differential equation Hayat et al. [6] applied the modified Adomian decomposition method with the Pade approximation and obtained the series solution. Rashidi [7] used the differential transform method with the Pade approximant and developed analytical solutions for this problem. In this paper we have used the Modified Homotopy analysis method to solve the non-linear boundary value problem.

2. Mathematical Formulation of the Problem:

Consider the magneto hydrodynamic flow of an incompressible viscous fluid over a stretching sheet at $y = 0$. The fluid is electrically conducting under the influence of an applied magnetic field $B(x)$ normal to the stretching sheet. The induced magnetic field is neglected. The resulting boundary layer flow is governed by the eqns. are as follows [6]

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (1)$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial y}=v\frac{\partial^2 u}{\partial y^2}-\sigma\frac{B^2(x)}{\rho}u \qquad (2)$$

Where we have u and v as the velocity components in the x - and y – directions respectively, n is the kinematic viscosity, ρ is the fluid density and σ is the electrical conductivity of the fluid. In equation (2), the external electric field and the polarization effects are negligible and in accordance with Chiam [8] the magnetic field $B(x)$ takes the form $B(x)=B_0 x^{n-1/2}$ and the associated boundary conditions according to [6] are as follows

$$u(x,0)=c\ x^n \quad , \quad v(x,0)=0$$
$$u(x,y)\to 0 \quad , \quad y\to\infty .$$

Where c and n are constants. On employing the following substitutions

$$t=\sqrt{\frac{c(n+1)}{2v}}x^{n-1/2}y$$
$$u=cx^n f'(t)$$
$$v=-\sqrt{\frac{c(n+1)}{2v}}x^{n-1/2}\left[f(t)+\frac{n-1}{n+1}tf'(t)\right]$$

and transforming eqns. (1) and (2) we get the governing equations as

$$f'''(t)+f(t)f''(t)-\beta(f'(t))^2-Mf'(t)=0 \qquad (3)$$

Subject to the boundary conditions

$$f(0)=0,\ f'(0)=1,\ f'(\infty)=0 \qquad (4)$$

Where $\beta=\frac{2n}{n+1}$ and $M=\frac{2\sigma B_0^2}{\rho c(1+n)}$

3. Analytical Expression for the Non-Linear Differential Equations (3) and (4) using the Modified Homotopy Analysis Method:

In this paper an analytic tool for nonlinear problems, namely the Homotopy analysis method (HAM) by Liao [9], is employed as our basic concept to solve the nonlinear differential eqns. (3) and (4). The Homotopy analysis method is based on a basic concept in topology, i.e. Homotopy by Hilton [10] which is widely applied in numerical techniques as in [11-14]. Unlike perturbation techniques [15-18], the Homotopy analysis method is independent of small/large parameters. Unlike all other reported perturbation and non-perturbation techniques such as the artificial small parameter method [19], the δ-expansion method [20] and Adomian's decomposition method [21], the Homotopy analysis method provides us with a simple way to adjust and control the convergence region and rate of approximation series. The homotopy analysis method has been successfully applied to many nonlinear problems such as viscous flows [22], heat transfer [23], nonlinear oscillations [24], nonlinear water waves [25], Thomas-Fermi's atom model [26], etc.

In particular, by means of the Homotopy analysis method, the author Liao [27] gave a drag formula for a sphere in a uniform stream, which agrees well with experimental results in a considerably larger region of Reynolds number than those of all reported analytic drag formulas. These successful applications of the Homotopy analysis method verify its validity for nonlinear problems in science and engineering. The Homotopy analysis method is a good technique comparing to other perturbation method. The Homotopy analysis method contains the auxiliary parameter h, which provides us with a simple way to adjust and control the convergence region of solution series.

In this paper we have used the Modified Homotopy analysis method and obtained the approximate analytical expression for the non-linear boundary value problem eqn. (3) with the boundary condition (4) (see Appendix B) as follows:

$$f(t)=\frac{1-\exp(-(\beta+M)t)}{\beta+M}-h\left(\begin{array}{l}-\dfrac{(\beta-1)\exp(-2(\beta+M)t)}{8(\beta+M)^3}-\dfrac{(M+1)\exp(-(\beta+M)t)}{(\beta+M)^3}\\ -\left(\dfrac{\beta-1}{4(\beta+M)^2}+\dfrac{M+1}{(\beta+M)^2}\right)t+\dfrac{M+1}{(\beta+M)^3}+\dfrac{\beta-1}{8(\beta+M)^3}\end{array}\right) \quad (6)$$

$$f'(t)=\exp(-(\beta+M)t)-h\left(\begin{array}{l}\dfrac{(M+1)\exp(-(\beta+M)t)}{(\beta+M)^2}\\ +\dfrac{(B-1)\exp(-2(\beta+M)t)}{4(\beta+M)^2}-\dfrac{M+1}{(\beta+M)^2}-\dfrac{\beta-1}{4(\beta+M)^2}\end{array}\right) \quad (7)$$

4. Results and Discussion:

The analytical solution for the nonlinear MHD boundary layer flow over a nonlinear stretching sheet has been obtained using the Modified Homotopy analysis method. The success of the procedure depends largely on how good our initial guess is. Also the selection of suitable values of h is very crucial to make our solution accurate. After choosing the appropriate values of the auxiliary parameter h we have given a graphical comparison between the non-dimensional stream function $f(t)$ versus the similarity variable t [Fig.1]. From which we can infer that the dimensionless stream function $f(t)$ decreases for increasing values of the magnetic parameter M for some fixed values of β. [Fig. 2] describes graphically the influence of non dimensional parameter β on the stream function $f(t)$ for some fixed values of M showing that $f(t)$ increases with increase in the values of β.

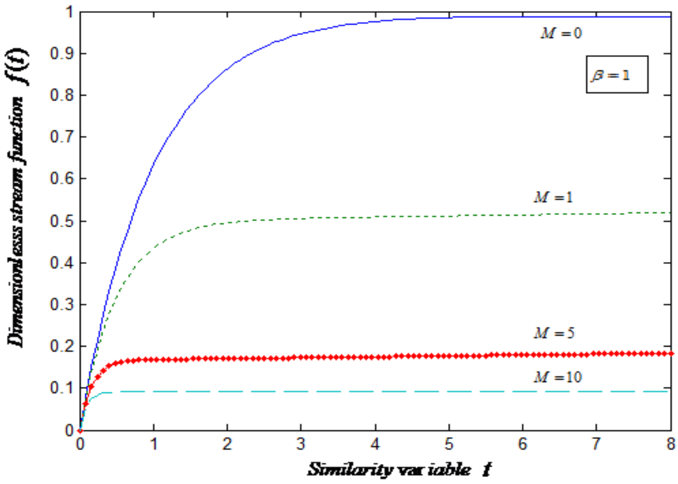

Figure (1): Dimensionless stream function $f(t)$ versus the similarity variable t. The curves are plotted for various values of the magnetic parameter M in some fixed value of the dimensionless parameter β for the equation (6)

However the effect of M and β on velocity needs attention for which we have plotted the velocity graphs [Fig. 3] and [Fig.4], the plot of dimensionless fluid velocity $f'(t)$ versus the similarity variable t which explains us that the dimensionless fluid velocity decreases with increasing values of M and β. It is noted that as magnetic interaction parameter M increases fluid velocity $f'(t)$ decreases elucidating the fact that the effect of magnetic field is to decelerate the velocity.

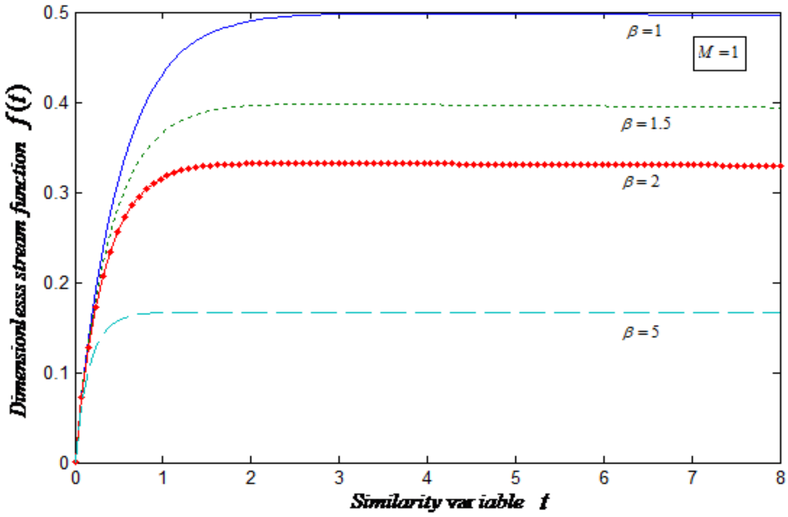

Figure (2): Dimensionless stream function $f(t)$ versus the similarity variable t. Here we have plotted the graph of equation (6) for various values of the dimensionless parameter β for a fixed magnetic parameter M.

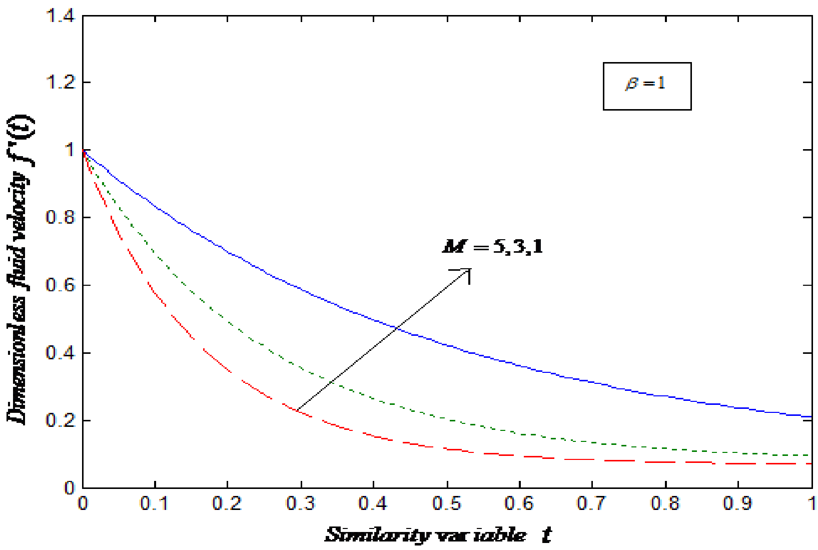

Figure (3): Dimensionless fluid velocity $f'(t)$ versus the similarity variable t. The graph of fluid velocity from the equation (7) for various values of the magnetic parameter M and a fixed dimensionless parameter β is obtained.

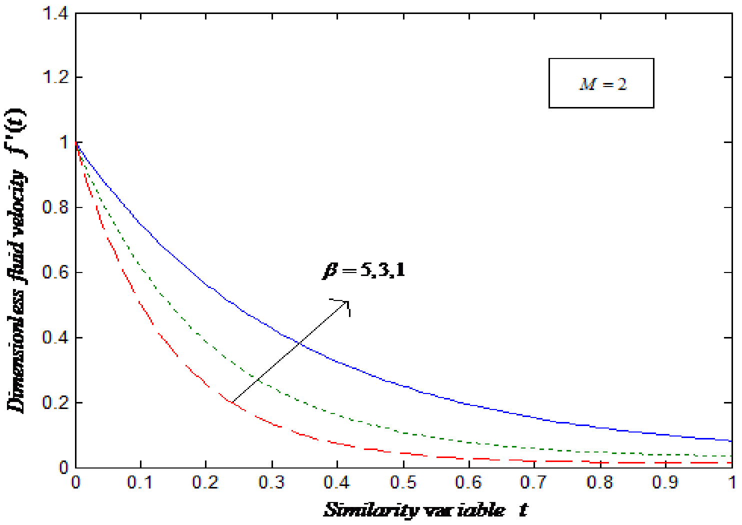

Figure (4): Dimensionless fluid velocity $f'(t)$ versus the similarity variable t. Using equation (7) graph is plotted for various values of the non dimensional parameter β and a fixed magnetic parameter M.

5. Conclusion:

In this paper the Modified Homotopy analysis method is employed to get the analytical solution for the MHD boundary layer flow over a non-linear stretching sheet. Using which we have obtained the solution as eqn. (6) for the dimensionless stream function $f(t)$ and on differentiating it we have obtained the expression for the fluid velocity as eqn. (7). Modified HAM is proved to be an easy and powerful tool for solving the non-linear problems. Through the graphical results thus obtained we have studied that the effect of the magnetic parameter M in the reduction of the velocity component is more than the non-dimensional parameter β. Thus the Modified HAM is proved to be the best for solving complicated MHD problems in fluid mechanics.

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Appendix: A

Basic Concept of the Homotopy Analysis Method:

Consider the following differential equation:

$$N[u(t)]=0 \qquad (A.1)$$

Where N is a nonlinear operator, t denotes an independent variable, $u(t)$ is an unknown function. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the conventional Homotopy method, Liao [28] constructed the so-called zero-order deformation equation as:

$$(1-p)L[\varphi(t;p)-u_0(t)]=phH(t)N[\varphi(t;p)] \qquad (A.2)$$

where $p\in[0,1]$ is the embedding parameter, $h \neq 0$ is a nonzero auxiliary parameter, $H(t) \neq 0$ is an auxiliary function, L an auxiliary linear operator, $u_0(t)$ is an initial guess of $u(t)$, $\varphi(t:p)$ is an unknown function. It is important to note that one has great freedom to choose auxiliary unknowns in HAM. Obviously, when $p=0$ and $p=1$, it holds:

$$\varphi(t;0)=u_0(t) \text{ and } \varphi(t;1)=u(t) \qquad (A.3)$$

respectively. Thus, as p increases from 0 to 1, the solution $\varphi(t;p)$ varies from the initial guess $u_0(t)$ to the solution $u(t)$. Expanding $\varphi(t;p)$ in Taylor series with respect to *p,* we have:

$$\varphi(t;p)=u_0(t)+\sum_{m=1}^{+\infty}u_m(t)p^m \qquad (A.4)$$

where

$$u_m(t)=\frac{1}{m!}\frac{\partial^m\varphi(t;p)}{\partial p^m}\Big|_{p=0} \qquad (A.5)$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter *h,* and the auxiliary function are so properly chosen, the series (A.4) converges at $p=1$ then we have:

$$u(t)=u_0(t)+\sum_{m=1}^{+\infty}u_m(t). \qquad (A.6)$$

Differentiating (A.2) for m times with respect to the embedding parameter p, and then setting p = 0 and finally dividing them by $m!$, we will have the so-called m *th*-order deformation equation as:

$$L[u_m-\chi_m u_{m-1}]=hH(t)\Re_m(\vec{u}_{m-1}) \qquad (A.7)$$

where

$$\Re_m(\vec{u}_{m-1})=\frac{1}{(m-1)!}\frac{\partial^{m-1}N[\varphi(t;p)]}{\partial p^{m-1}} \qquad (A.8)$$

and

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases} \quad \text{(A.9)}$$

Applying L^{-1} on both side of equation (A7), we get

$$u_m(t) = \chi_m u_{m-1}(t) + hL^{-1}[H(t)\Re_m(\vec{u}_{m-1})] \quad \text{(A10)}$$

In this way, it is easily to obtain u_m for $m \ge 1$, at M^{th} order, we have

$$u(t) = \sum_{m=0}^{M} u_m(t) \quad \text{(A.11)}$$

When $M \to +\infty$, we get an accurate approximation of the original equation (A.1). For the convergence of the above method we refer the reader to Liao [29]. If equation (A.1) admits unique solution, then this method will produce the unique solution.

Appendix: B

Analytical Expressions of the Equations. (3) and (4) using the Modified Homotopy Analysis Method:

This appendix contains the derivation of the analytical expression for $f(t)$ using the Modified Homotopy analysis method. Eqns. (3) and (4) are as follows

$$f''' + f\,f'' - \beta(f')^2 - M\,f' = 0 \quad \text{(B.1)}$$

Subject to the boundary conditions

$$f(0) = 0\ ,\ f'(0) = 1,\ f'(\infty) = 0 \quad \text{(B.2)}$$

We construct the Homotopy for these equations as follows,

$$(1-p)\left(f''' - M\,f'\right) = h\,p\left(f''' + f\,f'' - \beta(f')^2 - M\,f'\right) \quad \text{(B.3)}$$

The approximate solution for (B.3) is given by,

$$f = f_0 + p\,f_1 + p^2 f_2 + ... \quad \text{(B.4)}$$

By substituting the eqn. (B.4) into the eqn. (B.3) we get,

$$(1-p)\left(\frac{d^3}{dt^3}\left(f_0 + p\,f_1 + p^2\,f_2 + ...\right) - M\,\frac{d}{dt}\left(f_0 + p\,f_1 + p^2\,f_2 + ...\right)\right)$$

$$= h\,p\begin{pmatrix} \dfrac{d^3}{dt^3}\left(f_0 + p\,f_1 + p^2\,f_2 + ...\right) \\ + \left(f_0 + p\,f_1 + p^2\,f_2 + ...\right)\left(\dfrac{d^2}{dt^2}\left(f_0 + p\,f_1 + p^2\,f_2 + ...\right)\right) \\ - \beta\left(\dfrac{d}{dt}\left(f_0 + p\,f_1 + p^2\,f_2 + ...\right)\right)^2 - M\,\dfrac{d}{dt}\left(f_0 + p\,f_1 + p^2\,f_2 + ...\right) \end{pmatrix} \quad \text{(B.5)}$$

Now equating the coefficients of p^0 and p^1 we get the following equations,

$$p^0 :\ f_0 - M\,f_0^{'} = 0 \quad \text{(B.6)}$$

$$p^1 :\ f_1^{'''} - M\,f_1^{'} - f_0^{'''} - h\left(f_0^{'''} + f_0\,f_0^{''} - \beta\left(f_0^{'}\right)^2 - M\,f_0^{'}\right) = 0 \quad \text{(B.7)}$$

we have taken the initial guess solution to be

$$f_0 = \frac{1 - \exp(-(\beta + M)t)}{\beta + M} \quad \text{(B.8)}$$

The initial approximations are as follows:

$$f_0(0) = 0,\ f_0^{'}(0) = 1,\ f_0^{'}(\infty) = 0 \quad \text{(B.9)}$$

$$f_1(0)=0,\ f_1'(0)=0,\ f_1'(\infty)=0 \tag{B.10}$$

By solving eqns. (B.7) and eqn. (B.10) we get the following result

$$f_1 = h\left(\begin{array}{l}\dfrac{(\beta-1)\exp(-2(\beta+M)t)}{8(\beta+M)^3}+\dfrac{(M+1)\exp(-(\beta+M)t)}{(\beta+M)^3}\\ +\left(\dfrac{\beta-1}{4(\beta+M)^2}+\dfrac{M+1}{(\beta+M)^2}\right)t-\dfrac{M+1}{(\beta+M)^3}+\dfrac{\beta-1}{8(\beta+M)^3}\end{array}\right) \tag{B.11}$$

According to the Modified Homotopy analysis method we have

$$f=\lim_{p\to 1} f(t)=f_0+f_1 \tag{B.12}$$

Using the eqns. (B.8) and (B.11) in eqn. (B.12) we get the solution as in the text given by equation (6)

Appendix - C

Nomenclature:

Symbol	Meaning
$B(x)$	Variable applied magnetic induction.
β	Dimensionless parameter
M	Magnetic interaction parameter.
t	Similarity variable
$f'(t)$	Dimensionless fluid velocity
$f(t)$	Dimensionless stream function
u, v	Velocity component of fluid in x and y direction