International Journal of Engineering Research and Modern Education (IJERME)
ISSN (Online): 2455 - 4200
(www.rdmodernresearch.com) Volume I, Issue I, 2016

THE SYSTEM OF DIFFERENTIAL EQUATIONS IN LOTKA VOLTERRA MODEL

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Abstract:

In this paper, we build up a model to study the effects of predation on the population dynamics of the predator dynamics of the predators and its prey. More precisely, among the various predator-prey models, we focus on the Lotka Volterra model, which models two species system. This model can be used to simulate prey predator dynamics, and analyze when prey-predator populations are sustainable and when they are doomed, which can serve purposes like preventing species extinction. We illustrate example of prey-predator model and we obtain the solution.

Key Words: The Lotka Volterra Equations, Prey & Predator

The Lotka Volterra Model:

The lotka volterra equations ,also known as the predator –prey equations ,are a pair of first-order, non –linear, differential equations frequency used to describe the dynamics of biological systems in which two species interact ,one as a predator and the other as prey.

The populations change through time according to the pair of equations:

$$\frac{dx}{dt} = Ax - Bxy$$

$$\frac{dy}{dt} = -CY + Dxy$$

- x is the number of prey(for example : Deer)
- y is the number of some predators (for example, Tiger)
- $\frac{dy}{dt}$ and $\frac{dx}{dt}$ represent the growth rates of the two populations over time,
- t represent time, and
- A,B,C,D are positive real parameters describing the interaction of the two species:

  A - Growth rate of prey
  B - Searching efficiency or attack rate
  C - Predator mortality rate
  D - Growth rate of predator or predator's ability at turning food into offspring

Physical Meanings of the Equations:

The Lotka-Volterra model makes a number of assumptions about the environment and evolution of the predator and prey populations:

1. The prey population finds ample food at all times.
2. The food supply of the predator population depends entirely on the size of the prey population
3. The rate of change of population is proportional to its size.
4. During the process, the environment does not change in favor of one species and genetic adaptation is inconsequential.
5. Predators have limitless appetite.

The Lotka Volterra Equations:

Now that we have investigated a slightly more straightforward situation, we will work directly with the Lotka volterra equations. As we started earlier, there is no guarantee that a conserved quantity exists for a system, but assuming one does exist ,we will begin by finding the conserved quantity for the system,

$$\frac{dx}{dt} = Ax - Bxy$$

$$\frac{dy}{dt} = -Cy + Dxy$$

to start, multiply both sides of each equation by 1/xy to get

on the top yields

$$0 = \frac{dx}{dt}\frac{1}{xy} = \frac{A}{Y} - B$$

$$\frac{dy}{dt}\frac{1}{xy} = -\frac{C}{x} + D$$

Multiplying both sides of each equation by the appropriate derivative, we get

$$\frac{1}{xy}\frac{dx}{dt}\frac{dy}{dt} = \frac{dy}{dt}\left(\frac{A}{y} - B\right)$$

$$\frac{1}{xy}\frac{dx}{dt}\frac{dy}{dt} = \frac{dx}{dt}\left(-\frac{C}{x} + D\right)$$

Subtracting the bottom equation from the top yields

$$0 = \frac{dy}{dt}\left(\frac{A}{y} - B\right) + \frac{dx}{dt}\left(\frac{C}{x} - D\right)$$

Like before, $\frac{dE}{dt}$ = 0,so the right hand side must match our chain-rule expansion for dE/dt,

$$\frac{d}{dt}E(x,y) = \frac{dy}{dt}\frac{\partial}{\partial y}E(x,y)\frac{dx}{dt}\frac{\partial}{\partial x}E(x,y)$$

These two equations tell us that if,

$$\frac{\partial}{\partial y}E(x,y) = \frac{A}{y} - B$$

$$\frac{\partial}{\partial x}E(x,y) = \frac{C}{x} - D$$

then, like before, E (x(t),y(t)) is a constant as t varies . Thus we can write that

$$E(x,y) = \int\left(\frac{A}{y} - B\right)\partial y = \int\left(\frac{C}{x} - D\right)\partial x$$

Integrating we get,

$$E(x,y) = A\ln y - By + f(x) = C\ln x - Dx + g(y)$$

Again we see arbitrary functions of integration, which are eliminated when we take a partial derivative just like how ordinary integration introduces arbitrary constants, which ordinary derivatives wipe out. Merging the sides of the above equation as before, we get that

$$E(x,y) = A\ln y + C\ln y - By - Dx + M$$

with a little algebraic manipulation ,like before ,we see that

$$L - M = A\ln y + C\ln x - By - Dx$$

and so

$$E(x,y) = A\ln y + C\ln x - By - Dx = K$$

where K = L – M is a constant that depends on initial conditions and not on t.

Like we stated with the simpler system, this is not the only possible way to defined a conserved quantity for the Lotka – Volterra equations. For example, it is often the case that $e^{E(x,y)}$(with E(x,y) being the above version) is the chosen definition for the Lotka – Volterra equations conserved quantity ,which is perfectly valid since it will still always be constant as t varies.

However, since the way we have defined E(x,y) is also perfectly valid, and since it is a common version, we will stick to it for the remainder of our analysis.

Instead of straightforward polynomial functions ,we see natural logarithms, although there are linear terms as before .The reason for this difference stems from the differences in the initial equations : the simpler system had "-y" and "-x" where the Lotka – Volterra equation have " A/y" and " C/x"(compare the respective integrals in the derivations) . One thing to take from the conserved quantity for the Lotka Volterra equations is that, because of the natural logarithms, it is not defined anywhere where $x \le 0$ or $y \le 0$; hence, for any value of K,x and y will never go below 0 if the system starts out with E(x,y) = K (this matches our intuitive notion that there cannot be negative numbers of either predator or prey)

Example:

$$\frac{dx}{dt} = x(1-0.5y)=x-0.5xy = F(x,y)$$

And (1)

$$\frac{dy}{dt} = y(-0.75 + 0.25x) = -0.75y + 0.25xy = G(x,y)$$

For x and y positive

Solution:

The critical points of this system are the solutions of the algebraic equations

$$x(1-0.5y)=0 \quad (2)$$

$$y(-0.75 +0.25x)= 0 \quad (3)$$

namely, the points (0,0) and (3,2) fig (1) shows the critical points and a direction field for the system (1)

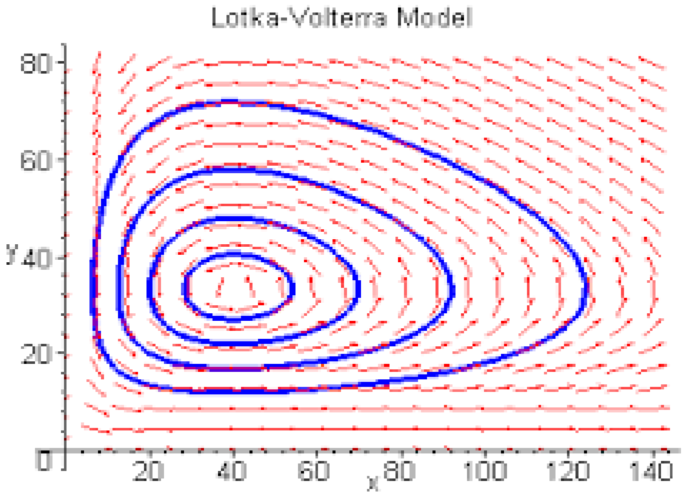

Figure (1): Critical points and direction field for the predator – prey system (1)

From this figure it appears that trajectories in the first quadrant encircle the critical point (2). Whether the trajectories are actually closed curves, or whether they slowly spiral in or out, cannot be definitely determined from the direction field. The origin appears to be a saddle point. The co ordinate axes are trajectories of Eqs (1). Consequently; no other trajectory can cross a co ordinate axis, which means that every solution starting in the first quadrant remains there for all time.

Next we examine the local behavior of solutions near each critical point. Near the origin we can neglect the nonlinear terms in Eqs (1) to obtain the corresponding linear system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -0.75 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{4}$$

The Eigen values and Eigen vectors of Eq (4) are

$$r_1 = 1,\ \xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix};\ r_2 = -0.75,\ \xi^{(2)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{5}$$

So its general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-0.75t} \tag{6}$$

Thus the origin is a saddle point both of the linear system (4) and of the nonlinear system (1) and therefore is unstable. One pair of trajectories enters the origin along the y axis all other trajectories depart from the neighborhood of the origin.
To examine the critical point (2) we can use the Jacobian Matrix

$$J = \begin{pmatrix} F_x(x,y) & F_y(x,y) \\ G_x(x,y) & G_y(x,y) \end{pmatrix} = \begin{pmatrix} 1 - 0.5y & -0.5x \\ 0.25y & -0.75 + 0.25x \end{pmatrix} \tag{7}$$

Evaluating J at the (2), we obtain the linear system

$$\frac{d}{du}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & -1.5 \\ 0.5 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{8}$$

Where u = x-3 and v=y-2.The Eigen values and eigenvectors of this system are

$$r_1 = \frac{\sqrt{3}i}{2}, \xi^1 = \begin{pmatrix} 1 \\ -i/\sqrt{3} \end{pmatrix}; r_2 = -\frac{\sqrt{3}i}{2}, \xi^2 = \begin{pmatrix} 1 \\ i/\sqrt{3} \end{pmatrix} \tag{9}$$

Since the Eigen values are imaginary, the critical point (2) is a center of the linear system (8) and is therefore a stable critical point for that system. This is one of the cases in which the behaviour of the linear system may or may not carry over to the nonlinear system, so the nature of the point (2) for the non-linear system (1) cannot be determined from this information.

The simplest way to find the trajectories of the linear system (8) is to divide the second of Eqs (8) by the first so as to obtain the differential equations.

$$\frac{dv}{du} = \left(\frac{dv}{dt}\right) / \left(\frac{du}{dt}\right) = (0.5u)/(-1.5v) = -u/3v$$

$$\text{Udu+3vdv} = 0 \tag{10}$$

Consequently,
$$u^2 + 3v^2 = k^2 \tag{11}$$

Where k is an arbitrary non negative constant of integration. Thus the trajectories of the linear system (8) are ellipse centered at the critical point and elongated somewhat in the horizontal direction.

Now let us return to the nonlinear system (1). Dividing the second of Eqs (1) by the first we obtain

$$\frac{dy}{dx} = \frac{y(-0.75+0.25x)}{x(1-0.5y} \qquad (12)$$

Equation (12) is a separable equation and can be put in the form

$$\frac{1-0.5y}{y}dy = \frac{-0.75+0.25x}{x}dx$$

From which if follows that

$$0.75 \ln x + \ln y - 0.5y - 0.25x = C \qquad (13)$$

Where C is a constant of integration. Although by using only elementary functions we cannot solve Equ (13) explicitly for either variable in terms of the other, it is possible to show that the graph of the equation for a fixed value of c is a closed curve surrounding the critical point (2).Thus the critical point is also a center of the nonlinear system (1) and the predator and prey populations exhibit a cyclic variation.

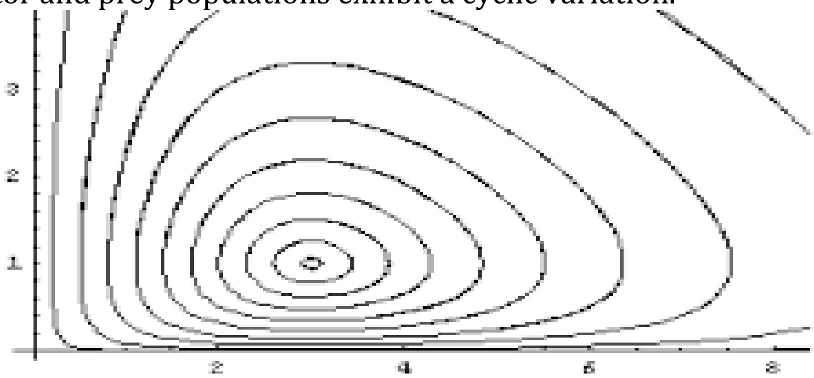

Figure (2): A phase portrait of the system (1)

Fig (2) shows a phase portrait of the system of equations. For some initial conditions the trajectory represents small variations in x and y about the critical point, and is almost elliptical in shape, as the linear analysis suggests. For other initial conditions the oscillations in x and y are more pronounced, and the shape of the trajectory is significantly different from an ellipse.

We observed that the trajectories are traversed in the counter clockwise direction. The dependence of x and y on for a typical set of initial conditions is shown in Fig (3)

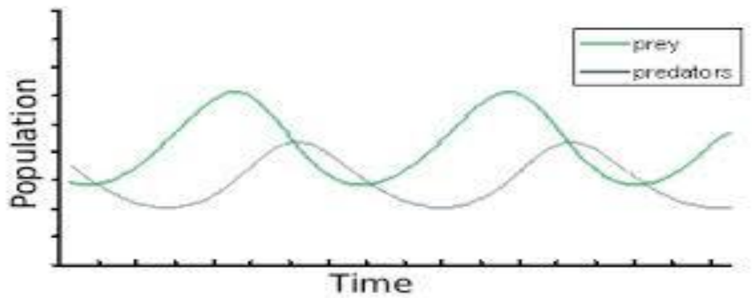

Figure (3): Variations of the prey and predator populations with time for the system (1)

We can note that x and y are periodic functions of t, as they must be since the trajectories are closed covers. Further the oscillation of the predator population lags behind that of the prey. Starting from a state in which both predation and prey populations are relatively small, the prey first increase because there is little predation. Then the predators with abundant food increase in population also. This causes heavier predation and the prey tends to decrease. Finally, with a diminished food supply, the predator population also decreases and the system returns to the original state.

Conclusion:

By starting with this model, one could simply account for one more variable, for instance hunting, and have an entirely new model. This would also help to increase the relevance of the science, because the more predation relationships this level of study can apply to, the more relevant the science becomes. Accounting for more variables can increase the adaptability of the model to increasingly more species. Other options for future work could include studying other predation relationships for which the Lotka Volterra Model may apply, particularly species which rely on one another as recourses while lacking substantial external variables. This model has simply broken the ground of endless possibility in the biological mathematics world.

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